Homework 3: Introduction to Probability

Lesson Review

Probability

Probability measures how likely an event is to occur. As such, in order to talk about the probability of an event, we must specify exactly the situation the event is occurring in. To illustrate, if I were to ask "What is the probability of getting 6?", you would have no way of answering. Am I rolling a dice? Drawing numbers from a hat? Maybe it's the chance I score 6 goals in a soccer game?

On the other hand, if I am specific about the situation, or **experiment**, my question about probability makes sense: "What is the probability of rolling a 6 on this 6-sided dice?". Since the experiment is exactly specified — rolling a dice — you could confidently answer $\frac{1}{6}$.

There are cases where the experiment is well specified, so we can ask probabilistic questions, yet be unable to answer them perfectly: "What is the probability of the US men's hockey team beating the Finland hockey team in the next winter olympics?" While the experiment in this case is specified exactly, there are so many factors, or **variables**, involved in this question that we would only be able to approximate an answer.

However, for the work we are doing, we will always be able to calculate the exact answer. So, how do we calculate the probability of an **event** occurring? To do that, we first need to understand exactly what probability means. The probability of an event is the number of ways the experiment can yield that event, divided by the total number of possible **outcomes** for the experiment.

$$P(\text{event}) = \frac{\text{number of outcomes for the event}}{\text{total number of outcomes for experiment}}$$

For example, if we roll a standard 6-sided dice, there are 6 possible outcomes: I could roll a 1, 2, 3, 4, 5, or 6; and each number is just as likely to be rolled as any of the others. As such, there are a total of 6 possible outcomes for the experiment of rolling a 6-sided dice. If we were to ask, "What is the probability of rolling a 5?", well there is only one outcome for that — rolling a 5, so the probability would be $\frac{1}{6}$. If I were instead ask, "What is the probability of rolling an even number?", then the answer would be $\frac{3}{6} = \frac{1}{2}$ because there are 3 possible outcomes that lead to the event: I could roll a 2, 4, or a 6.

If I were to instead ask, "What is the probability of rolling a 7 on this 6-sided dice?", then the answer would be 0. Why? Because there are 0 outcomes that result in me rolling a 7, so the probability is $\frac{0}{6} = 0$. On the other end, I could ask, "What is the probability of rolling a positive number on this 6-sided dice?", then the answer would be 1. Why? Because all 6 outcomes result in me rolling a positive number, so the probability is $\frac{6}{6} = 1$.

Looking closely, the probability for an event will always be a number between 0 and 1. Why? Well if the event cannot occur, such as, "What is the probability of drawing 5 aces from this deck of cards?", then the answer is always 0, since regardless of how many possible outcomes the experiment has, the number of outcomes for the event is 0, so we will have

$$P(\text{event}) = \frac{0}{\text{total number of outcomes}} = 0$$

Similarly, the highest probability of any event would occur when every outcome matches that event, so we would have

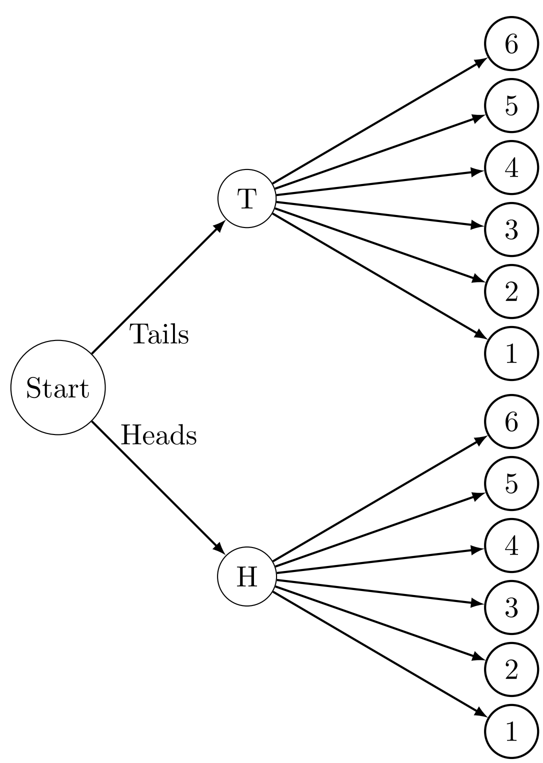
$$P(\text{event}) = \frac{\text{all outcomes match}}{\text{total number of outcomes}} = \frac{x}{x} = 1$$

All values between 0 and 1 (exclusive) would mean that the event can occur; however, it is not guaranteed to occur if we ran the experiment. For example, drawing the 2 of clubs from a deck of cards could occur with a probability of $\frac{1}{52}$; however, it is not guaranteed to occur — I could just as easily draw the king of hearts. As the probability of an event gets closer to 1, then it is more likely for that event to occur if we were to run the experiment. On the other hand, when the probability is closer to 0, then it is less likely to occur.

Complex Experiments

We can run experiments that have multiple steps, for example, we could flip a coin and then roll a 6-sided dice. We could then ask questions like, "What is the probability I flip heads, and roll an odd number?", or "What is the probability I flip tails, and roll a number less than 3?".

Just as with our simple experiments, we calculate probability in the same way: we need to find the number of outcomes that match the event, and divide by the total number of outcomes. How can we determine the total number of outcomes? Well we can represent our experiment as a tree-diagram:



The circles, or **nodes**, in our tree-diagram, represent the outcome of a step in our experiment. For example, the **H** node represents flipping heads, and the **T** node represents flipping tails. From the heads node, we then split off 6 more branches, leading to six, **leaf nodes**, which represent each possible outcome of rolling our dice. Each of these twelve leaf nodes represents a possible outcome of our experiment, and they represent all possible outcomes as a result of how we constructed our tree-diagram.

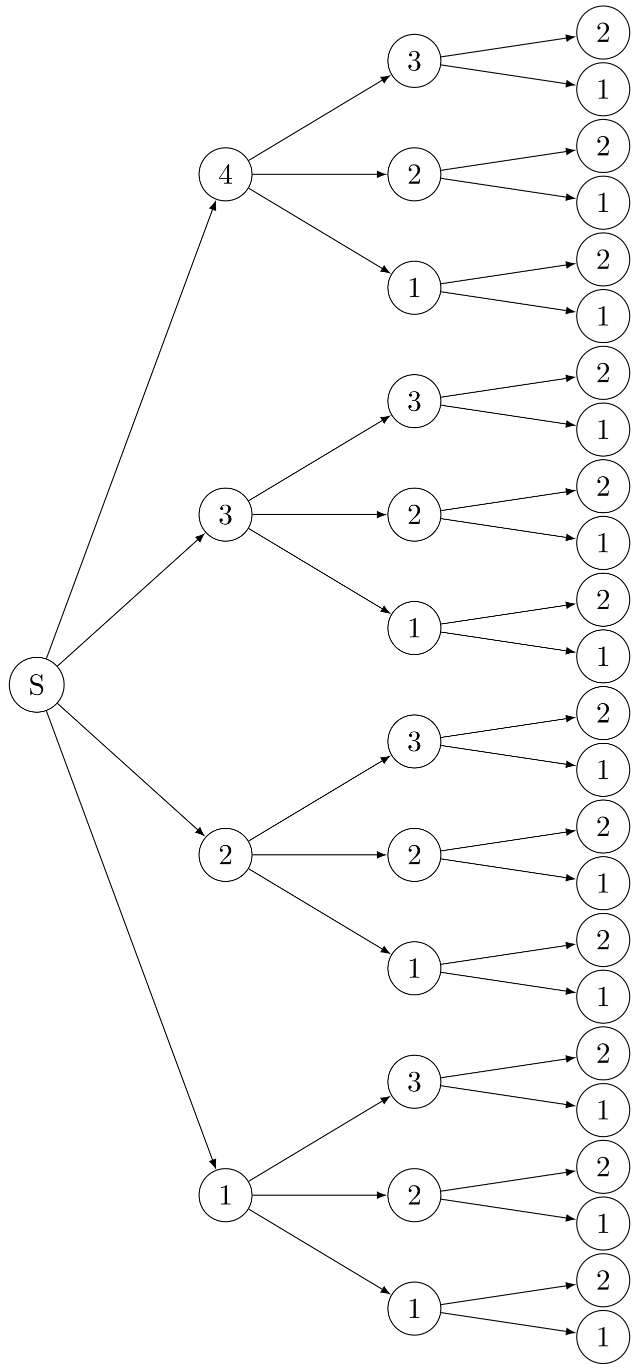
Since each leaf-node represents a single outcome, it is now trivial to answer our questions: "What is the probability I flip heads, and roll an odd number?" Well, there are 3 outcomes, $(\mathrm{H}, 1), (\mathrm{H}, 3), (\mathrm{H}, 5)$, so the probability would be $\frac{3}{12} = \frac{1}{4}$.

Similarly, "What is the probability I flip tails, and roll a number less than 3?" There are only 2 outcomes this time, $(\mathrm{T}, 1), (\mathrm{T}, 2)$, so the probability would be $\frac{2}{12} = \frac{1}{6}$.

Example

You will roll a 4-sided dice, then a 3-sided dice, then a 2-sided dice. What is the probability of rolling all odd numbers?

Let us again make a tree-diagram — this one will be quite large. . .



The tree has $4 \times 3 \times 2 = 24$ total outcomes. To find the probability of rolling all odd numbers, we count leaf nodes reached via only odd branches. The odd values on the 4-sided die are $\{1, 3\}$, on the 3-sided die are $\{1, 3\}$, and on the 2-sided die is $\{1\}$, giving $2 \times 2 \times 1 = 4$ favorable outcomes.

$$P(\text{all odd}) = \frac{4}{24} = \frac{1}{6}$$

Determining Outcomes without Diagrams

As we can see with our last example, drawing a tree-diagram to answer probabilistic questions is not always practical. For example, I would never be able to fit the tree for the experiment of rolling a 10-sided dice 100 times on a single piece of paper. So what can we do instead? While the tree-diagram is a useful mental model to have, we can compute the total number of outcomes directly for many experiments.

In our last example, we rolled a 4, then a 3, and finally a 2 sided dice. If we look at our tree-diagram, we first split into 4 nodes to represent the 4 possible outcomes of rolling the 4-sided dice. From each of those nodes, we split into 3 nodes for the outcomes of rolling the 3-sided dice, and finally, from each of those 12 nodes, we split into 2 more nodes for the outcomes of the 2 sided dice.

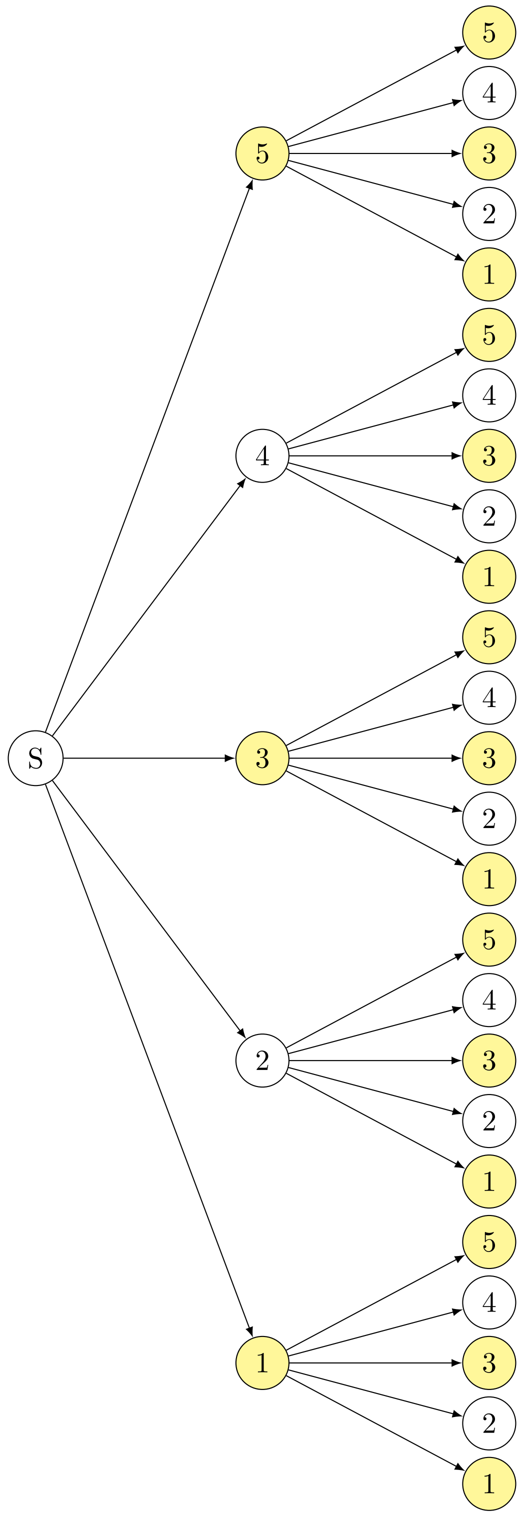
As such, starting from the top: we have 4 things each containing 3 things — in our case, 4 nodes each splitting out into 3 additional nodes. That's just basic multiplication: $4 \times 3 = 12$, and we do in fact have 12 nodes as children to our first 4 nodes. Similarly, each of those 12 things contains 2 more things — in our case, these 12 intermediate nodes each split into 2 additional nodes to represent our final dice roll. Again, this is just multiplication: $12 \times 2 = 24$. Similarly, we could also look at it as we have 4 buckets, each containing 3 smaller buckets, each containing 2 things — once again, we know how to deal with this problem via multiplication: $4 \times 3 \times 2 = 24$.

In general, if we are running a multi-step experiment where each step has a fixed number of possible outcomes that are equally likely to occur (such as a dice roll), then the total number of outcomes will be the product of the number of outcomes for each step. For example, if our experiment has 5 steps, and the first step has 3 outcomes, the second 4, the third 2, the fourth 3, and the fifth 10, then the total number of possible outcomes would be $3 \times 4 \times 2 \times 3 \times 10 = 720$.

Counting Outcomes for an Event

We can leverage this same multiplication rule to find the number of outcomes for an event in a multi-step experiment if we know the number of outcomes for each step in the event. For example, if we are rolling a 5-sided dice four times, and want to know "what is the probability of rolling all odd numbers?", we can instead find the number of outcomes where we roll an odd number on a 5-sided dice if we were to only roll it a single time, then leverage the same multiplication rule...

Why is that? Let's draw the two layers of this tree-diagram to illustrate



In this diagram, we have highlighted the nodes that represent odd numbered rolls. Just as the nodes formed buckets allowing us to multiply when counting all outcomes, we can again use the highlighted nodes to form our buckets. This time though, each bucket will only contain three of the possible outcomes (the odd numbered rolls) though. As such, applying the multiplication rule, we can see the total number of ways we could roll all odd numbers would be $3 \times 3 \times 3 \times 3 = 81$. We can also apply the multiplication rule to find the total number of possible outcomes: since there are 5 possible outcomes for rolling the dice once, then there are $5 \times 5 \times 5 \times 5 = 625$ total possible outcomes. As such, the probability of rolling all odd numbers is $\frac{81}{625}$.

Practice Problems

Part 1: Single-Step Probability

For each problem, identify the favorable outcomes and calculate the probability.

1. You roll a 12-sided die (numbered 1–12). What is the probability of rolling a number divisible by 3?

2. You roll a 10-sided die (numbered 1–10). What is the probability of rolling a prime number?

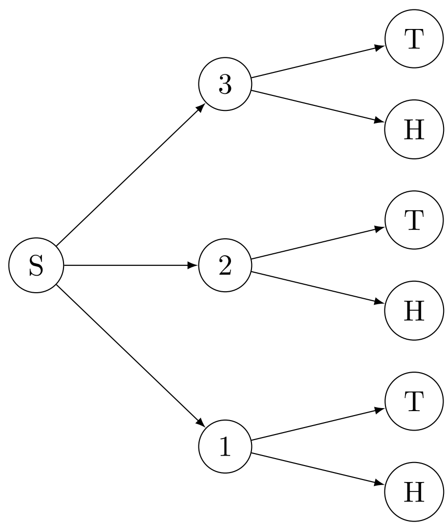
3. You roll a 20-sided die (numbered 1–20). What is the probability of rolling a perfect square?

4. You roll a 9-sided die (numbered 1–9). What is the probability of rolling a factor of 36?

5. You roll a standard 6-sided die. What is the probability of rolling a number that is both even *and* greater than 2?

Part 2: Multi-Step Experiments (Tree Diagram Provided)

6. You roll a 3-sided die (numbered 1–3), then flip a fair coin.



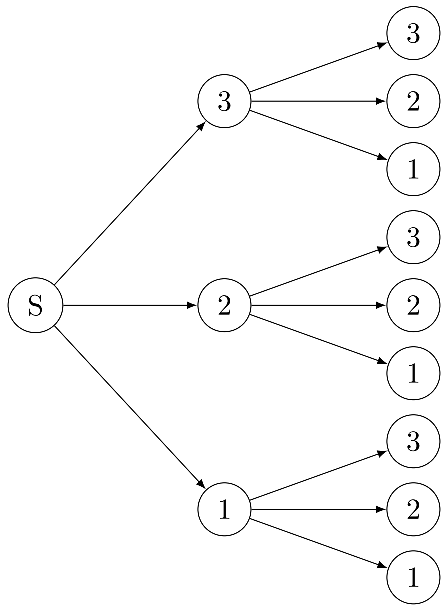
(a) How many total outcomes are there?

(b) What is the probability of rolling a 2 on the die?

(c) What is the probability of rolling an odd number and flipping heads?

(d) What is the probability of flipping tails?

7. You roll a 3-sided die (numbered 1–3) twice.



(a) How many total outcomes are there?

(b) What is the probability of rolling the same number on both dice?

(c) What is the probability that the two dice sum to 4?

(d) What is the probability that at least one die shows a 3?

Part 3: Multi-Step Experiments (Draw the Tree)

8.

You flip a fair coin, then roll a 4-sided die (numbered 1–4).

(a) Draw the tree diagram for this experiment in the space below.

(b) How many total outcomes are there?

(c) What is the probability of flipping heads and rolling a 4?

(d) What is the probability of flipping tails and rolling an even number?

(e) What is the probability of rolling a number less than 3?

9.

You roll a 3-sided die (numbered 1–3), then roll a 4-sided die (numbered 1–4).

(a) Draw the tree diagram for this experiment in the space below.

(b) How many total outcomes are there?

(c) What is the probability that the first die shows an odd number and the second die shows an even number?

(d) What is the probability that the first die shows a 2?

(e) What is the probability that the two dice sum to 5?

Part 4: Challenge

10. You are playing a medieval fantasy game as a mighty hero fighting an evil wizard. The game is **turn-based**: each turn, both you and the wizard attack once. A **5-sided die** is rolled at the start of each turn to decide who attacks first — an **odd** roll means *you* attack first, an **even** roll means the *wizard* attacks first.

To attack, you roll a **6-sided die**; the result is the damage you deal to the wizard. The wizard rolls a **5-sided die** to determine the damage he deals to you. Both players start with **10 HP**. The last one standing wins.

Example game:

Turn	Event	Your HP	Wizard HP
Start	—	10	10
Turn 1	Turn die: 3 (odd) $\Rightarrow$ you attack first		
	Your attack roll: 5	10	5
	Wizard attack roll: 4	6	5
Turn 2	Turn die: 2 (even) $\Rightarrow$ wizard attacks first		
	Wizard attack roll: 5	1	5
	Your attack roll: 2	1	3
Turn 3	Turn die: 5 (odd) $\Rightarrow$ you attack first		
	Your attack roll: 4	1	-1 **(You win!)**

(a)

Draw a tree diagram for all possible outcomes of a single turn. *Hint: there are three dice rolls in a turn — the turn die, your attack, and the wizard's attack.*

(b) On a single turn, what is the probability that you attack first, deal more than 3 damage to the wizard, and take less than 3 damage from the wizard?

(c)

Suppose a turn ends with you at 6 HP and the wizard at 4 HP. Draw a tree diagram showing all possible outcomes of the *next* turn, marking which outcomes result in you defeating the wizard (i.e. you deal ≥ 4 damage before he can kill you).

(d) Based on your tree diagram from (c), what is the probability that you defeat the wizard on that turn?