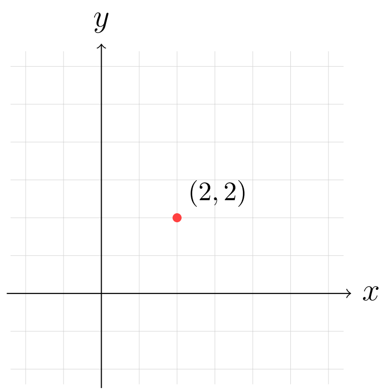
Intro to Geometry

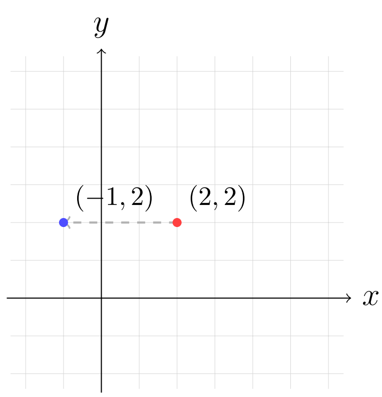
Space, Points, and Coordinates

The study of geometry can be thought of as the study of objects in **space**. But what "space" are we talking about? Like most everything in math, it depends on the context. In general we characterize a space by how many **dimensions** it has: there is one-dimensional space (1D) - this is equivalent to a number line. There's two-dimensional space (2D) - this is equivalent to an infinite plane. And of course, there's the space we exist in: three-dimensional space (3D). We can actually go beyond 3D space to four-dimensional, five-dimensional, or even 999-dimensional space; however, our ability to mentally visualize these higher-dimensional spaces is quite challenging.

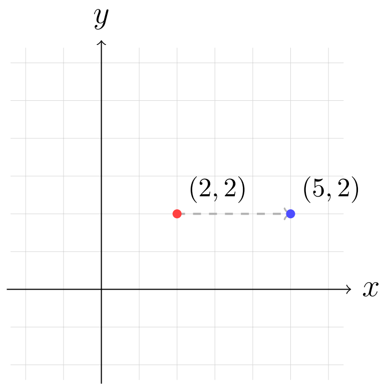
You might be wondering: "What determines the dimensionality of a space?" and "If we can't picture higher-dimensional space, then how can we work with it?" - both excellent questions, with related answers. Let us start with the first question. The dimensionality of a space is determined by the number of **orthogonal** components the space has (the independent directions). But what does that even mean?! Let's explain using 2D space - 2D space is an infinite plane, which we typically visualize with the **Cartesian Plane** seen below with a **point** graphed on it:



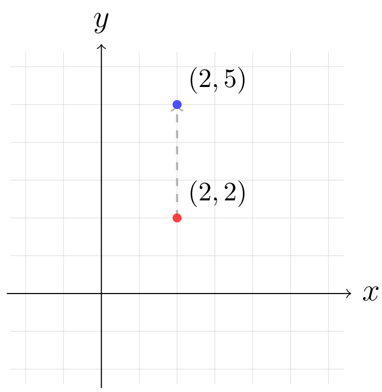
If we were to shift our point to the left by 3, our image would become (the original point is shown in red, the shifted point in blue):



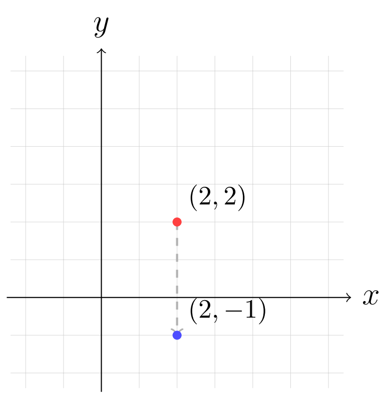
Similarly, we could instead shift it to the right by 3:



Or up by three:



Or down by three:



Notice that when we shift the point left or right, the y-coordinate does not change. Similarly, when we shift the point up or down, the x-coordinate does not change. That's because left/right movements are orthogonal to up/down movements, and vice-versa. In other words: we can move the point up/down all we want, and it will never change how left/right the point is; or we can move the point left/right all we want, and that will never change how up/down the point is - left/right movement is *independent* of up/down movement and vice-versa.

Now let's get a little more formal: as we can see in our above example of moving the point - we were able to label the location of the point with its **coordinates**. In space, we can talk about any specific location by referring to its coordinates. To specify the coordinates of a location, we refer to its position in each direction (for each orthogonal component). For example, in 2D space, we write $(4, -7)$ to refer to the position that is 4 units to the right of the center, and 7 units below the center. Traditionally, in 2D space, the left/right direction is assigned to the variable x, and the up/down direction to y - so our coordinates are of the form (x, y) for some real numbers x and y.

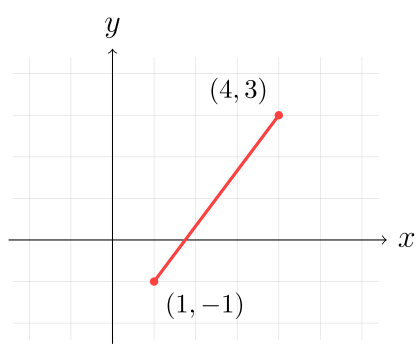
Similarly, in 3D space, we stick with the variables x and y; furthermore, we describe the relative height of an object using the variable z. As such, in 3D space, our coordinates are of the form: (x, y, z).

Extending this concept to higher-dimensional spaces, for example, 6D space, our coordinates would just have 6 entries: (x, y, z, a, b, c). Each entry represents the relative position from the center for one of the orthogonal components that make up the space. In general, for N-dimensional space, the coordinates would have N entries. As such, while we may not be able to visualize what 6, or 12, or 99-dimensional space looks like with a graph, we are still able to talk about these higher-dimensional spaces using their coordinates.

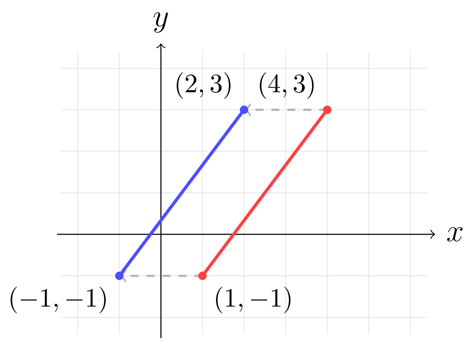
Line Segments

In our above examples, we had drawn a point in 2D space and labeled its location with its coordinates. A point in space can be thought of as the most basic element of space. However, a single point in space is not all that interesting by itself. As such, we could plot two different points and connect them with a straight line to form a **line segment**. Line segments, like points, are basic objects in space, so it is natural for us to start our study with them.

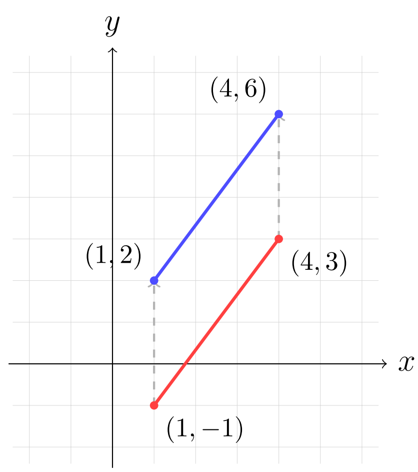
Since all line segments are created by drawing a line between two points, we can describe any line segment by specifying what those two points are - these are called the **endpoints**. For example, we can talk about the line segment that has endpoints $(1, -1)$ and $(4, 3)$ and we can draw it on our plane:



We can then move the line segment to the left by 2 as seen in the graph below:



If we look at the new endpoints, we'll notice that we subtracted 2 from the x-coordinate of each endpoint to give us the new endpoints $(-1, -1)$ and $(2, 3)$. This is not a coincidence - our coordinates and graphs are just different perspectives of the same object: our line segment. As such, what do you think would happen if we instead added 3 to the y-coordinate? Our endpoints would then be $(1, 2)$ and $(4, 6)$. Let's graph this new line segment alongside the original on the same graph:



As we can see, this just moved the line segment up the y-axis by 3, the same type of behavior we saw when we adjusted the x-coordinate.

Movement as a Function

What if we wanted to describe movement of a line segment using a function? Can we create a function that moves any line segment up by 3, while maintaining its position relative to the y-axis?

Recall from our previous lesson that we need three components for a function: an input, a rule to apply to that input, and an output. Since our function needs to move a line segment, then both the input and output to the function would be a line segment. That just leaves us with creating a rule that maps our original line segment into a new one. Let's start by describing that in plain English: "Add three to the y-coordinate of each endpoint."

Great, it looks like we have a function! However, let's see if we can write our function more like an equation. In that case, we are going to need a variable to describe the input to our function; let's call the input line segment L. But wait, our function rule is to add 3 to the y-coordinate of each endpoint - so how can we get the y-coordinates of the endpoints of L? We can just use variables again! Since L is just some line segment, we know that it has two endpoints, so we can name them with variables. Let's call our endpoints e_0 and e_1. We're getting closer, we now have two endpoints, but we don't yet have their y-coordinates.

What if we just applied the same logic again? Since e_0 and e_1 are points in space, we know that they each have coordinates. We may not know exactly what their coordinates are, so what do we do? We name them with variables, of course. As such, we can say that the coordinates of e_0 are (x_0, y_0) and the coordinates of e_1 are (x_1, y_1). For convenience, we can write $e_0 = (x_0, y_0)$ and $e_1 = (x_1, y_1)$. Similarly, we can write $L = \overline{e_0e_1}$.

Finally, we have the y-coordinates, so we should now be able to write our function as an equation. Let's call our function "up":

$$\mathrm{up}(L) = \mathrm{up}(\overline{e_0e_1}) = \overline{e_0'e_1'} \text{ such that } e_0' = (x_0, y_0 + 3) \text{ and } e_1' = (x_1, y_1 + 3)$$

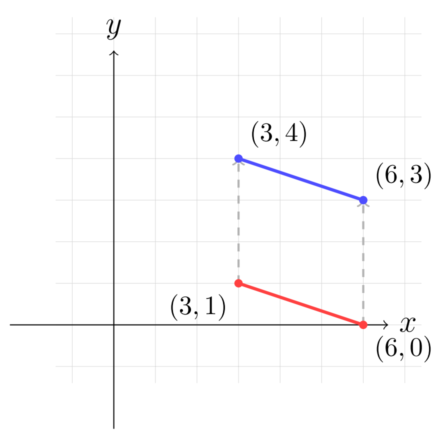
Although, this is still a bit clunky. Since a line segment is determined by its two endpoints, and endpoints are just coordinates, let's allow ourselves to write our line segment in coordinate form. For example, we can write $L = \{(x_0, y_0), (x_1, y_1)\}$ to say that L is the line segment that has endpoints with coordinates (x_0, y_0) and (x_1, y_1). For example, the line segment with endpoints $(3, 1)$ and $(6, 0)$ can be written as $\{(3, 1), (6, 0)\}$. Now let's rewrite our equation for our function using our new format:

$$\mathrm{up}(L) = \mathrm{up}(\{(x_0, y_0), (x_1, y_1)\}) = \{(x_0, y_0 + 3), (x_1, y_1 + 3)\}$$

Now let's plug our example line segment $\{(3, 1), (6, 0)\}$ into our function to get a new line segment:

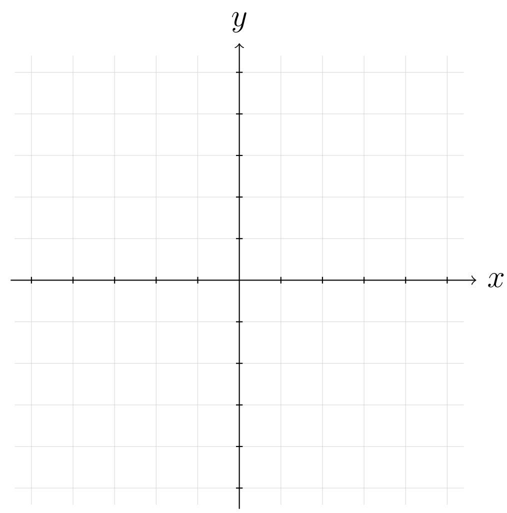
$$\mathrm{up}(\{(3,1),\ (6,\ 0)\}) = \{(3, 1 + 3), (6, 0 + 3)\} = \{(3, 4), (6, 3)\}$$

Graphing our result, we can see that our function works as expected:

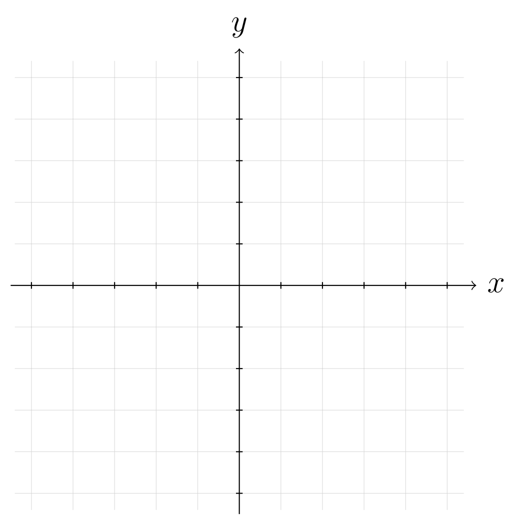


Practice Problems

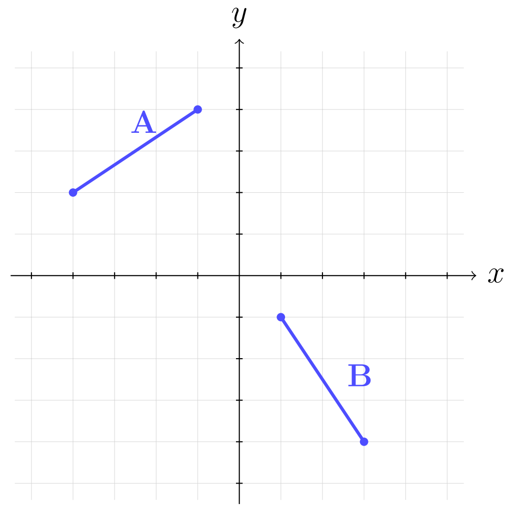
1. Graph and label the following points: $(3, 2)$, $(-4, 1)$, $(0, -3)$, $(-2, -2)$, and $(4, -1)$.



2. Graph and label the following line segments: $\{(-3, 2), (1, 4)\}$, $\{(2, -1), (4, 3)\}$, and $\{(-4, -2), (-1, -4)\}$



3. Write each of the graphed line segments below in coordinate form: $L = \{(x_0, y_0), (x_1, y_1)\}$



4. We can write functions to map one point to another in the format $f(x, y) = (x', y')$. For example, the function that moves a single point to the left by 1 would be $f(x, y) = (x - 1, y)$.

(a) Create a function that moves a point to the right by 5.

(b) Create a function that moves a point down by 2.

(c) Create a function that moves a point up by 1.

(d) **(Challenge)** Create a function that moves a point across the x-axis. For example, if the point was located above the x-axis, then the function would move the point below the x-axis. Similarly, if the point was located below the x-axis, then the function would move it above the x-axis.

5. We can also write functions to map one line segment to another in the format $f(\{(x_0, y_0), (x_1, y_1)\}) = \{(x_0', y_0'), (x_1', y_1')\}$. For example, the function that moves a line segment to the left by 1 would be $f(\{(x_0, y_0), (x_1, y_1)\}) = \{(x_0 - 1, y_0), (x_1 - 1, y_1)\}$.

(a) Create a function that moves a line segment to the right by 5.

(b) Create a function that moves a line segment down by 2.

(c) Create a function that moves a line segment up by 1.

(d) **(Challenge)** Create a function that moves a line segment across the x-axis. For example, if the line segment was located above the x-axis, then the function would move it below the x-axis. Similarly, if the line segment was located below the x-axis, then the function would move it above the x-axis. For this question, you can assume that the line segment does not intersect the x-axis.