Introduction to Functions

Lesson

In your study of mathematics, you will often be working with **functions**. You will be asked to calculate the value of functions at specific inputs, to solve for variables using systems of functions, to graph functions - the list goes on and on. Since so much of math centers around functions, we should take some time to make sure we have mastered the basics.

You've probably seen functions quite a bit before - things like $y = 4x + 5$, $f(z) = z^2$, and $g(x, y) = f(x) + h(x)$. But what exactly is a function? A function is a **rule** that converts an **input** into an **output**. But that's pretty generic, what do we mean by rule? What can our inputs be? What about our outputs? Rather than trying to answer these questions directly, let's look at some examples.

Example

"For every real number, add one" is a function - its input is any real number, its rule is to add one to the input, and its output is also a real number. We could also choose to write this function as an equation: $h(z) = z + 1, z \in \mathbb{R}$. The $h(z) = z + 1$ tells us our rule, while the $z \in \mathbb{R}$ part tells us that our input is any real number (the $\in$ symbol means that the variable belongs to the set, in this case, the real numbers, $\mathbb{R}$). Now that we have it written down, we can use it as we please: $h(5) = 5 + 1 = 6$, $h(101.434) = 101.434 + 1 = 102.434$.

Example

"For every positive integer, return its remainder when divided by 2" is also a function - its input is any positive integer, its rule is to find the integer's remainder when divided by 2. So what will its output be? Well, there are only two possible outcomes when we divide an integer by 2: it's either an even number, and thus would have a remainder of 0; or it's an odd number, and thus would have a remainder of 1. As such, our output is one of $\{0, 1\}$. We can also write this as an equation, let $n \in \mathbb{N}$:

$$\text{even}(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

With our function, we can now write things like $\text{even}(57) = 1$, and $\text{even}(10112) = 0$.

Example

"For any polygon, return the number of sides" is yet another function. Interestingly, this function's input is not a number, but a shape - a polygon to be specific. However, that's perfectly okay! After all, nowhere did we say that the input to a function had to be a number. Furthermore, we clearly have a well-defined rule - just count the sides of the polygon, giving us a number for an output. Can we be more specific about our output? Well, there's no such thing as a polygon with a fractional side - so our output will always be a whole number. Furthermore, the polygon with the least number of sides is a triangle - so our output will always be a whole number greater than or equal to 3. We can also write this as an equation, let p be any polygon, define the function $\text{sides}(p) = $ the number of sides of p. If p were a parallelogram, then $\text{sides}(p) = 4$; if p were an octagon, then $\text{sides}(p) = 8$.

Example

"Mix together two primary colors" - for example, mixing red and blue gives purple. Is this still a function? The input would be two primary colors - we can represent that as an **ordered pair** $(\text{color}_1, \text{color}_2)$ such that color_1 and color_2 are variables for our two colors. When we mix two primary colors, we always get a secondary color, so our output will be a secondary color. As such, it seems we have everything for a function: an input, rule, and output - so mixing primary colors is in fact a function. As such, we can call it $\text{mix}(\text{color}_1, \text{color}_2)$, and then write things like $\text{mix}(\text{blue}, \text{yellow}) = \text{green}$.

Domain and Range

Looking at our above examples, we can see that they have a few things in common. When it comes to the input for a function, we specify a set of values that our function works with - this is often called the **domain** of the function. Furthermore, for each input in our domain, we have a rule that assigns it to some output value. If we were to apply the rule to every possible input in our domain, we would then have every possible output our function can produce. The set of all possible outputs of a function is called its **range**.

Practice Problems

1. "For any real number, square it."

(a) Write the sentence as a function, $f(x)$.

(b) What is the domain of the function?

(c) What is the range of the function?

(d) Calculate $f(5)$, $f(-1)$, and $f(10)$.

2. "For any real number, multiply it by 2 and then add 5."

(a) Write the sentence as a function, $f(x)$.

(b) What is the domain of the function?

(c) What is the range of the function?

(d) Calculate $f(3)$, $f(-1.5)$, and $f(0)$.

3. "For any positive integer, return its remainder when divided by 3."

(a) What is the domain of the function?

(b) What is the range of the function?

(c) Calculate $f(8)$, $f(15)$, and $f(22)$.

4. "For any two primary colors, mix them together to get a new color." *(The primary colors are red, blue, and yellow.)*

(a) What is the domain of the function?

(b) What is the range of the function?

(c) Calculate $f(\text{red}, \text{yellow})$, $f(\text{blue}, \text{yellow})$, and $f(\text{red}, \text{blue})$.