Exponents

What Is an Exponent?

One of the most powerful tools in mathematics is to eliminate complexity through notation. By abstracting away complexity, we allow ourselves to see the same problem from a whole new perspective, often revealing new insights that were hidden from us before. One of the very first examples a student comes across is the multiplication symbol

$$a \times b = \underbrace{a + a + \cdots + a}_{b \text{ copies}}.$$

Imagine a world where we didn't have it: what's

$$7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 + 7$$

Did you have to slow down and count all the 7's? I did, and I'm writing this lesson! What if I instead asked you what 7×9 is? With just one simple symbol, $\times$, we make reasoning about repeated addition far easier on ourselves. In some ways, we allow ourselves to forget that multiplication is really just addition: we turn the trees into a forest.

In today's lesson, we will introduce exponents to encode the concept of repeated multiplication. Of course, multiplication is just repeated addition, so does that make exponents repeated, repeated addition? Repeated addition that's repeated? As you can see, from the perspective of exponents, the concept of addition gets quite challenging to reason about (addition that's repeated, repeatedly? That doesn't sound right either). Thankfully, since we can effectively forget that multiplication is repeated addition, we can easily say that exponents are repeated multiplication, in the same way that multiplication is just repeated addition.

The notation for exponents is a bit different from multiplication's $\times$ symbol, but it behaves in just the same way. To express

$$\underbrace{7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7}_{9 \text{ copies}}$$

with exponents, we write 7^9. More generally, to express multiplying n 7s together, we write 7^n:

$$7^n = \underbrace{7 \times 7 \times \cdots \times 7}_{n \text{ copies}}$$

When using exponent notation, we call 7 the **base** and n is the **exponent**.

> **Exponent.** An **exponent** tells us how many times a number is repeatedly multiplied. In the expression b^n, the **base** b is the number being multiplied, and the exponent n

tells us how many copies of b to include in the product:

$$b^n = \underbrace{b \times b \times \cdots \times b}_{n \text{ copies}}.$$

Often, we will say that b is **raised to the power of** n.

Example. To evaluate 2^4, we multiply together four copies of our base 2:

$$2^4 = 2 \times 2 \times 2 \times 2 = 4 \times 2 \times 2 = 8 \times 2 = 16$$

The base is 2 and the exponent is 4. Notice that 2^4 is *not* $2 \times 4 = 8$ — a common first mistake.

Example. To evaluate 3^3, we multiply together three copies of our base 3:

$$3^3 = 3 \times 3 \times 3 = 9 \times 3 = 27$$

Similarly, for 6^2, we multiply our base 6 by itself:

$$6^2 = 6 \times 6 = 36$$

Now that we understand the basics, let's take a look at what happens when we raise the bases 1 through 10 to the powers of 1 through 5, as seen in the below table.

base $= b$	b^1	b^2	b^3	b^4	b^5
1	1	1	1	1	1
2	2	4	8	16	32
3	3	9	27	81	243
4	4	16	64	256	1,024
5	5	25	125	625	3,125
6	6	36	216	1,296	7,776
7	7	49	343	2,401	16,807
8	8	64	512	4,096	32,768
9	9	81	729	6,561	59,049
10	10	100	1,000	10,000	100,000

Reviewing our table, we can easily read off the result of raising each base to each power. There are several points of interest to notice as we do so:

- The first row is all 1, since $1 \times 1 = 1$ and thus $1^n = 1$.
- The last row just adds a trailing 0 to each entry, with the number of 0s being the exponent on 10; e.g. $10^3 = 1{,}000$, which has three 0s.
- The first column is just the base itself, since $b^1 = b$ for every base; e.g. $7^1 = 7$.

- Unlike multiplication and addition, we quickly run into very large numbers with very small inputs; e.g. $9^4 = 6{,}561$ versus $9 \times 4 = 36$ and $9 + 4 = 13$.

The Product Rule

With our basic understanding in place, it's time to turn our attention to trying to understand what rules exponents follow. In order to do so, we will have to remind ourselves what exponents really are: repeated multiplication. As such, to figure out how exponents work, we will explore using the rules of multiplication. To start, let us return to our example of

$$7^9 = \underbrace{7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7}_{\text{9 copies}}$$

If we were to calculate this by hand, showing each multiplication as a single step, then our next line would be:

$$7^9 = 49 \times \underbrace{7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7}_{\text{7 copies}}$$

and our next:

$$7^9 = 343 \times \underbrace{7 \times 7 \times 7 \times 7 \times 7 \times 7}_{\text{6 copies}}$$

While we could continue in this manner, let us instead stop to notice $49 = 7^2$ and $343 = 7^3$, so these two lines are really just equal to:

$$7^9 = 7^2 \times \underbrace{7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7}_{\text{7 copies}}$$

and:

$$7^9 = 7^3 \times \underbrace{7 \times 7 \times 7 \times 7 \times 7 \times 7}_{\text{6 copies}}$$

but wait, we can apply that same thinking to the first line since $7^1 = 7$ and thus:

$$7^9 = 7^1 \times \underbrace{7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7}_{\text{8 copies}}$$

Now look closely at the right halves of these equations. Each one is nothing but a pile of 7s multiplied together — and that is exactly what a power of 7 is. So can we just swap in 7^8, 7^7 and 7^6 for them and be done?

We can, but it is worth slowing down for a moment to see why we are allowed to. Written out with no parentheses anywhere, a line like our second one is one long chain of multiplication:

$$7^9 = 49 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7$$

Nothing in that chain tells us which multiplication to do first. We are used to working left to right, so we would multiply 49×7, then multiply that by the next 7, and so on. However, as we already know, multiplication is **associative**: for any three numbers, $(a\times b)\times c = a\times(b\times c)$; for example:

$$(2 \times 3) \times 5 = 6 \times 5 = 30 \qquad \text{and} \qquad 2 \times (3 \times 5) = 2 \times 15 = 30.$$

Using the associative rule for our example, we can instead start multiplying the right end of 7s, leaving the 49 alone:

$$49 \times 7 \times 7 \times \cdots \times 7 = 49 \times \left(\underbrace{7 \times 7 \times \cdots \times 7}_{7 \text{ copies}} \right) = 49 \times 7^7$$

Doing the same substitution to each of our lines, we see that:

$$\begin{aligned} 7^9 &= 7^1 \times 7^8 \\ 7^9 &= 7^2 \times 7^7 \\ 7^9 &= 7^3 \times 7^6 \end{aligned}$$

And if we continue in this manner, we also see:

$$\begin{aligned} 7^9 &= 7^4 \times 7^5 \\ 7^9 &= 7^5 \times 7^4 \\ 7^9 &= 7^6 \times 7^3 \\ 7^9 &= 7^7 \times 7^2 \\ 7^9 &= 7^8 \times 7^1 \end{aligned}$$

As we can see, $7^9 = 7^x \times 7^y$ when $x + y = 9$. This is not a special property of 7; if we performed this same exercise with any base, we would see the same result. More generally,

> **Product Rule.** The **product rule** tells us that for any base b, and any positive integers x and y, that $b^{x+y} = b^x \times b^y$. For example, $123^5 = 123^{2+3} = 123^2 \times 123^3$.

Practice Problems

Part 1: Reading and Evaluating Exponents

1. For each expression below, name the **base** and name the **exponent**, and then write out what the expression means as repeated multiplication. Do *not* evaluate it. For example, for 3^4 you would write:

$$\text{base } 3, \quad \text{exponent } 4, \quad 3^4 = 3 \times 3 \times 3 \times 3$$

(a) 5^3

(b) 2^7

(c) 10^4

(d) 8^1

(e) 6^5

(f) 12^2

2. Each of the following is an expression read aloud, the way we say it out loud rather than the way we write it. Write each one in exponent notation, and then evaluate it. For example, "four raised to the power of two" becomes:

$$4^2 = 4 \times 4 = 16$$

(a) five raised to the power of three

(b) two raised to the power of four

(c) ten raised to the power of two

(d) nine raised to the power of one

(e) three raised to the power of four

(f) six raised to the power of two

3. Evaluate each expression by writing it out as repeated multiplication. Show *every* line of your work, multiplying only one pair of numbers per step, so that each line has one fewer copy of the base than the line before it. For example:

$$3^4 = 3 \times 3 \times 3 \times 3 = 9 \times 3 \times 3 = 27 \times 3 = 81$$

(a) 3^2

(b) 10^3

(c) 2^5

(d) 5^3

(e) 4^4

(f) 6^3

(g) 9^2

(h) 7^3

Part 2: The Product Rule

4. Use the product rule to evaluate each expression. Split the exponent into a product of two powers, substitute the two values you are given, and then multiply. For example, if you were told that $5^1 = 5$ and $5^2 = 25$:

$$5^{1+2} = 5^1 \times 5^2 = 5 \times 25 = 125$$

(a) 2^{2+3}, where $2^2 = 4$ and $2^3 = 8$

(b) 3^{1+3}, where $3^1 = 3$ and $3^3 = 27$

(c) 10^{2+3}, where $10^2 = 100$ and $10^3 = 1{,}000$

(d) 7^{1+3}, where $7^1 = 7$ and $7^3 = 343$

(e) 4^{2+2}, where $4^2 = 16$

(f) 6^{2+1}, where $6^2 = 36$ and $6^1 = 6$

(g) 9^{3+1}, where $9^3 = 729$ and $9^1 = 9$

(h) 5^{1+4}, where $5^1 = 5$ and $5^4 = 625$

Every answer above is a power you can look up in our table — once you have finished, check each one against it.